The Restoration of Extended Astronomical Images using the Spatially-Variant Point Spread Function

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Abstract
In this paper we address the problem of image restoration of extended astronomical objects, where the effects of inhomogeneous random media, such as atmospheric turbulence, result in distortion over the pupil plane. Distortions are usually expressed in terms of the point spread function (PSF) and can be used to restore an image with the application of a deconvolution algorithm. Since the PSF is generally spatially dependent, restoration of extended images within specific homogeneous (isoplanatic) regions is required. Inconsistencies at the borders of each region result in error boundaries between restored isoplanatic regions and the object. To minimise such distortion, we determine the optimal size of each isoplanatic region and then apply a linear filter to estimate the PSF between boundaries.

Keywords: Image restoration, deconvolution, astronomical imaging, point-spread function.

1 Introduction
Astronomical images can include point source objects, i.e. stars, or extended objects, such as the Moon, planets, nebula, or star clusters. In addition to the ever-present effect of noise from imaging equipment and optical defects from instrumentation, images from ground-based telescopes are distorted by wavefront aberrations caused by atmospheric turbulence. The spatial impulse response function, commonly referred to as the point spread function (PSF), is used to represent such distortions, and can either be applied over the entire image, or within regions uniquely defined by the isoplanatic angle [1]. The combination of such regions forms an extended image, where the spatially variant PSF (SVPSF) is used for image restoration.

Wide-field images, distorted by the effects of an inhomogeneous medium, can be segmented into areas referred to in the literature as isoplanatic regions or patches [2]. This work considers image restoration of extended objects over a wide field-of-view (FOV), where the effects of a perturbing medium, such as atmospheric turbulence, is the cause of anisoplanatic distortion. Within each isoplanatic region, the PSF remains relatively constant. However, the PSF is spatially variant over the image plane. We assume that an algorithm to predict the spatially-variant PSF (SVPSF) [3], [4], is available for image restoration. This is a necessary prior, since it is not possible to directly image a point-source over an image of an extended object, such as a planet or the Moon. Once the SVPSF has been determined, a deconvolution algorithm is then used to restore the image.

Estimation of the SVPSF has employed parameterisation of the PSF, based on the metrics of the image [5], [6]. A modal approach using Zernike polynomials [7] has been used, based on deconvolution from wavefront sensing (DWFS) [3]. We use a similar strategy; however our estimate is based on a temporal method employing a recurrent artificial neural network to predict the SVPSF [8].

Section 2 provides a brief background on the image model and the PSF. Section 3 discusses how the image model is used to provide data for restoration. Section 4 describes how the isoplanatic region is determined and this is followed by a discussion on image restoration in Section 5. An improved image restoration method, based on a spatially variant image model, including our results, are given in Section 6. Lastly, Section 7 concludes this paper and outlines future work.

2 The Image Model
In this section the relationship between the wavefront aberrations and the wavefront phase is defined. This is followed by a description of the spatially invariant PSF (SIPSF) and SVPSF; each are then used to describe the image model. A more thorough discussion on spatially variant im-
age models is provided by Andrews & Hunt [6] and Banham & Katsaggelos [9].

An aberrated wavefront, \( W(x, y) \), is defined as its deformation, in terms of a spherically shaped wavefront. The phase fluctuation \( \phi(x, y) \) is of primary interest in this paper and can be expressed as a function of the wavefront aberration, \( W(x, y) \), by

\[
\phi(x, y) = \frac{2\pi}{\lambda} W(x, y),
\]

where \( \lambda \) is the wavelength of light and \((x, y)\) are the spatial co-ordinates of the imaging system.

Atmospheric turbulence is the primary cause of wavefront aberrations and can be observed in an optical instrument, such as a telescope supporting a charge-coupled device (CCD) camera, as intensity variations in the image plane. A curvature wavefront sensor [10] is used to convert intensity patterns into linear phase data, \( \phi(x, y) \), as seen in the exit pupil, \( P(x, y) \). Such phase fluctuations are related to the PSF, \( h(x, y) \), by [11]

\[
h(x, y) = \left\| \text{FT}\left\{ P(x, y) \exp[-j\phi(x, y)] \right\} \right\|^2,
\]

where \( \text{FT} \) is the Fourier transform operator.

### 2.1 The Spatially-Invariant PSF

An image taken of an extended object can appear blurred if the object and the surroundings, w.r.t. the image plane, are moved during exposure. Such degradations are described as motion blur [12] and are typified by camera movement. Such distortions are invariant over the entire image plane, i.e., not a function of the spatial image plane, and thus, the PSF is described as being spatially-invariant (SIPSF).

Spatial invariance of the PSF is a defining parameter for isoplanatic regions, however in practice, the PSF is subject to minor changes. The SIPSF forms a block (2D) Toeplitz matrix, such that each descending diagonal is a constant [6].

The degraded image, \( d_r(x, y) \), can be expressed as a convolution between the PSF, \( h(x, y; k, l) \), and the object, \( f(x, y) \), and represented as

\[
d_r(x, y) = \sum_{k, l \in \Gamma} f(k, l) h(x-k; y-l) + \eta(x, y),
\]

where \( \eta(x, y) \) is the noise term and \( \Gamma \) defines the isoplanatic region. In the case of motion blur, \( \Gamma \) is the entire image, however in most imaging applications, particularly concerning astronomical imaging, the SVPSF is required for restoration.

### 2.2 The Spatially-Variant PSF

Given the PSF, \( h(x, y) \) defined in (2), the generalised image model can be expressed as [6]

\[
d_{r+}(x, y) = \sum_{k, l \in \Gamma_+} f(k, l) h(x, y; k, l) + \eta(x, y),
\]

where \( h(x, y; k, l) \) can represent either the SIPSF or SVPSF, \( d_{r+}(x, y) \) is the intensity of the pixels in the degraded image, \( f(k, l) \) is the original image, \( \Gamma_+ \) is the region that defines an anisoplanatic patch, and \( \eta(x, y) \) is observation noise, measured as a Gaussian process of mean zero and variance \( \sigma^2 \).

The SVPSF is used when the PSF can explore image space as a function of the aberration and is explicitly defined by, and confined to, a specific region within object space.

Due to the inhomogeneous nature of turbulence and the chaotic effects of wavefront aberration over the pupil and corresponding image plane, the SVPSF is of particular interest in our research. For example, \( \Gamma_+ \) in (4) corresponds to \( N \times M \) isoplanatic regions, each bound by region, \( \Gamma \). Thus, a generalised image can be defined within an anisoplanatic region as

\[
\Gamma_+ = \{ \Gamma_{1,1}, \Gamma_{1,2}, \cdots \Gamma_{2,1}, \Gamma_{2,2}, \cdots \Gamma_{M,N} \}.
\]

### 2.3 Zernike Polynomials

Zernike polynomials are a set of 2D orthonormal basis functions that can be used to represent wavefront aberrations. They represent the statistical eigenfunctions of optical distortions that quantitatively classify each aberration using a set of polynomials. Typically expressed in polar coordinates, Zernike polynomials, \( Z_i(\rho, \theta) \), as defined by Noll [7], are sets of functions or modes that are orthogonal over the unit circle, where \( \rho \) is the radius, \( 0 \leq \rho \leq 1 \), and \( \theta \) is the angle a point makes with the \( x \)-axis.

Based on this representation, the linear combination of \( K \) aberrations over a unit circle of radius \( R \) results in an approximation of the phase perturbation

\[
\phi(R\rho, \theta) \approx \sum_{i=2}^{K} a_i Z_i(\rho, \theta),
\]

where \( \rho \) is the normalised aperture given an aperture of radius, \( R \), and \( i \) is the Zernike mode. Lower-order aberrations, \( Z_2, Z_3 \) commonly referred to in the literature as tilt, result in the displacement of the PSF. Higher order aberrations, \( Z_3 \cdots Z_K \), each result in a specific aberration of the PSF, e.g. \( Z_4 \) is attributed to defocus, and \( Z_5 \), astigmatism. When \( K = \infty \), (6) is an exact representation of the phase in the exit pupil. Additionally, the piston term, \( Z_1 \), has been removed in Eqn. 6, as is common for single aperture instruments.
3 Simulation Environment

Direct measurement of the SVPSF of an extended object, such as the Moon is not possible due to the absence of point sources. However, several methods have been proposed to determine the SVPSF. These include parameterisation [13] [3], modal tomography [14], and more recently, prediction using recurrent neural networks [8], using a spatio-temporal image model [4].

The isoplanatic region, comprisingcoefficients, is represented by an aberrated SIPSF and defined an angle, $\theta_0$, as it subtends a portion of the FOV from the centre of the point source.

The perturbed wavefront from each source was represented by an aberrated SIPSF and defined an anisoplanatic region. Each SIPSF comprised coefficients, $a_i$ in (6). A scaled version of a set of four SIPSFs, dominated by low-order aberrations that result in motion blur, is shown in Figure 1.

$$\theta_0 = 58.1 \times 10^{-3} \lambda^{6/5} \left[ \int_0^L C_n^2(z) z^{5/3} dz \right]^{-3/5},$$ (7)

where $\lambda$ is the optical wavelength, $z$ is the altitude, $C_n^2$ is the structure constant of the turbulence, and $L$ is the path length through turbulence.

In order to quantify the error induced by a SIPSF as it explores image space, a simulation was conducted using a phase screen discussed in Section 3. A region on the image plane was associated with a spatially variant PSF. An $N \times N$ isoplanatic sub-region, comprising a reference SIPSF, $h_{Ref}(\cdot)$, and centred within a $X \times Y$ anisoplanatic region, was used as the basis for comparison. As the angular separation and orientation of each SIPSF to the reference SIPSF was varied, a 2D array of SIPSFs, $h(\cdot)$, was calculated. Each calculation used $N \times N$ sized regions and was based on localised phase distortions, $\phi(x, y)$. As the SIPSF, $h(\cdot)$, explored the anisoplanatic region, the mean square error (MSE), based on the reference SIPSF, $h_{Ref}(\cdot)$, was calculated. These data resulted in a 2D error function over the anisoplanatic region, $\epsilon_{MSE}$. Collectively, the region, $\Gamma_+$, formed the domain of the SVPSF.

The expression used to estimate the MSE of the SVPSF over $\Gamma_+$, is given by

$$\epsilon_{MSE}(x, y) = \frac{1}{N^2} \sum_{k,l \in \Gamma} \left[ h(k + x, l + y) - h_{Ref}(k + X/2, l + Y/2) \right]^2.$$

A 3D simulation of the spatially varying PSF is shown in Figure 2. The significance of this simulation demonstrates that for severe degradations, even the smallest deviation from the centroid of each SIPSF can result in significant errors that correspond to image distortion after deconvolution. Given our simulation model, the isoplanatic angle, $\theta_0$, was estimated to be 10 $\mu$ rads. This result was in agreement with Roggemann and Welsh [1].

5 Image Restoration

The forward image model given by (3) defines the image, $d_{\Gamma_+}(x, y)$, as the convolution between the object, $f(x, y)$, and the SVPSF, $h(x, y; l, k)$. To
can be restored by the deconvolution of the PSF, monly referred to as

restoring the sub-images that comprised isoplanatic regions, defined by \( 2 \times 2 \) sized blocks [13]. Thus, a continuous image can be reconstructed using a tiled or mosaic approach. Our method interpolates each adjacent set of SIPSFs to construct a new SIPSF that is valid for sub-quadrants over adjacent, isoplanatic boundaries.

6 Proposed Method & Results

Equation (5) is used to define a set of SIPSFs for the spatially variant PSF, \( \Gamma \). Each SIPSF represents the distortion within a specific isoplanatic sub-region, \( \Gamma \). Equation (7) states that the size of the isoplanatic region is dependent on the severity of the distortion, measured in terms of \( C^2 \).
Table 1: Image Reconstruction Comparisons

<table>
<thead>
<tr>
<th>SVPSF Image Comparison of Fig. 4</th>
<th>Metric</th>
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<tbody>
<tr>
<td></td>
<td>MSE</td>
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<tr>
<td>Tiled SIPSF, (a) &amp; (c)</td>
<td>14.54</td>
</tr>
<tr>
<td>Interpolated SIPSF, (a) &amp; (d)</td>
<td>13.86</td>
</tr>
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</table>

the wavelength of light, $\lambda$, and the height of the turbulence, $z$. Once the size of each region has been determined, the SIPSF is used to reconstruct a portion of the image. However, since each SIPSF is arranged in the form of a mosaic, the borders surrounding each isoplanatic region exhibited higher MSE.

To minimise this distortion, we interpolate the phase components associated with the SIPSF from each adjacent isoplanatic region. Each interpolated PSF is used in conjunction with image data from a corresponding boundary portion of each sub-image. The L-R-EM algorithm is then used to reconstruct the boundary portion between each sub-image. The resulting sub-image, or strip, is aligned with each sub-image pair to minimise discontinuities between isoplanatic regions.

To test our method we used a high-resolution image of the Moon [21], shown in Figure 4 (a), and convolved each quadrant with respective SIPSFs shown in Figure 1. Figure 4 (b) shows a distorted image with additive noise, created using the simulation platform discussed in Section 3. We then individually deconvolved each distorted region using the algorithm discussed in Section 5.1 and repositioned each quadrant; a detailed image of the residual distortion at one of the boundaries is shown in Figure 4 (c). Lastly, to minimise residual distortion, we applied linear interpolation to the Zernike coefficients and constructed a new SIPSF that was used to deconvolve each boundary region. A detail showing reduced distortion at two boundary regions is shown in Figure 4 (d).

The mean-squared error (MSE) was our initial metric used to compare the image quality of the reconstructed image in Figure 4 (c), with the portion of the original image shown in Figure 4 (a). We then compared the original image to our post-processed image in Figure 4 (d). The results showed a small improvement, in terms of MSE, as listed in Table 1. However, a second similarity metric (SIMM) was used that provided a method for assessing the perceptual qualities of the image [22]. This method showed a considerable improvement, where a perfect match is equated to unity. Our comparative results are listed in Table 1.

7 Conclusion

Anisoplanatic imaging of extended objects over a wide-FOV has been discussed in this paper. A method for extended image restoration, employing the SVPSF and simulation platform, has been proposed. The method relies on point sources, separated by $2\theta_0$ to act as references to estimate distortion over individual isoplanatic regions. Reconstruction is performed using the SIPSF over isoplanatic regions, however a linear estimator is used to interpolate the PSF within sub-regions, thereby improving restoration, and both methods are assessed using two metrics. Further work on parameterising the PSF is planned, using image sequences of the Jovian Planets and their respective satellites as point sources.

References

Figure 4: (a) Original extended object, (b) Aberrated noisy image, (c) Deconvolved image using four SIPSFs, (d) Deconvolved image with linear filtering.


